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THE UNIFORM APPROXIMATION PROPERTY IN ORLICZ SPACES

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ABSTRACT

It is proved that for every reflexive Orlicz space X there is a function $n(k,\varepsilon)$ so that whenever E is a k-dimensional subspace of X there exists an operator $T: X \to X$ such that $T_{|E} = \text{identity}$, $\|T\| \leqq 1+\varepsilon$ and $\dim TX \leqq n(k,\varepsilon)$. Some general facts concerning the uniform approximation property are also presented.

Introduction

The bounded approximation property (b.a.p.) is, as well known, shared by the common separable Banach spaces and in particular by all the spaces having a Schauder basis. It is usually quite easy to verify that a given concrete space has this property even without using a basis. The property which corresponds to the b.a.p. in the local theory of Banach space is called the uniform approximation property (u.a.p.). This property was introduced by A. Pelczynski and H. P. Rosenthal [11].

DEFINITION. *A Banach space X is said to have the uniform approximation property if there is a $\lambda > 1$ and a function $n(k)$ so that whenever E is a k-dimensional subspace of X there is an operator $T: E \to E$ for which $Tx = x$; $x \in E$ (i.e. $T_{|E} = \text{identity}$), $\|T\| \leqq \lambda$ and $\dim TX \leqq n(k)$.*

When the particular values of λ or $n(k)$ are of importance we shall say that X has the λ-u.a.p. or even the $(\lambda, n(k))$-u.a.p. If the T in the definition can be chosen so that it is also a projection we shall say that X has the uniform projection property.

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To check whether a given concrete Banach space has the u.a.p. seems to be much harder than checking the b.a.p. In [11] it was shown that all the L_p spaces have the u.a.p. (even the uniform projection property). The first step of the proof of this assertion gives some information for general Banach lattices, but the second part of this proof (which is trivial in the setting of L_p) uses a property which actually characterizes the L_p-spaces and thus works only in this setting. The verification of the validity of u.a.p. in other concrete lattices seems to require a much more detailed analysis. Just finding e.g. a Schauder basis in the space is not enough. It was observed by W. B. Johnson (cf. [11]) that the existence of a space which fails to have the b.a.p. implies easily the existence of a space with a Schauder basis which fails to have u.a.p. Recently, A. Szankowski [13] proved that the existence of an unconditional or even a symmetric basis does not ensure the u.a.p.

Before continuing let us make a brief comment concerning the interest in studying the u.a.p. We have already mentioned that it seems to us to be the natural approximation property in the local theory of Banach spaces. The u.a.p. is certainly of interest in connection with other properties studied in Banach space theory; in particular, of course, the various global approximation properties. For example, it follows from Theorem 3 below, that in order to verify that X^{**} has the b.a.p. it is enough to verify that X has the u.a.p. This may be a useful remark in cases where X is a "relatively small" space while X^{**} is a large non-separable space. It is also quite likely that the u.a.p. will play a role in approximation theory. The approximation property originated from the study of the question whether every compact operator $S: X \to Y$ is the limit in operator norm of finite rank operators. The u.a.p. comes into play if we ask questions of the following type: Given S as above and $\varepsilon > 0$, for which integer k does there exist a $T: X \to Y$ with $\dim TX \leqq k$ and $\| T - S \| \leqq \varepsilon$? (The answer depends on the parameters appearing in the u.a.p. for X or Y and on the degree of compactness of S, i.e. on the metric entropy of the image under S of the unit ball of X.)

In Section 2 below we prove some general results concerning the u.a.p. Theorem 1 states that if X has the λ-u.a.p. for some $\lambda > 1$ and if X is superreflexive then X has the λ-u.a.p. for every $\lambda > 1$. This theorem can be viewed as a local version of a result of Grothendieck [3] stating that reflexive spaces with the approximation property have already the metric approximation property. The proof is however entirely different. From Theorem 1 it follows easily that a superreflexive space X has the u.a.p. iff X^* has this property (Theorem 2 below). We do not know whether Theorems 1 and 2 remain valid without the assumption of superreflexivity. Without this assumption we have only the following result (Theorem 3 below): A Banach space X has the u.a.p. (if and) only if X^{**} has this property. (The "if" part is of course trivial.) This result emphasizes again the local nature of the u.a.p. and also exhibits an interesting difference between this property and the b.a.p.

In Section 3 we pass from general spaces to Orlicz spaces (in our context it makes no difference if we consider Orlicz sequence or function spaces). Our main result (Theorem 4) shows that the reflexive Orlicz spaces have the u.a.p. We give an explicit construction of the operators T (as a matter of fact projections) which appear in the definition of the u.a.p. The explicit construction does not, however, give operators of norm arbitrarily close to 1. We get a bound λ depending on the space (or more precisely on the Δ_2 constants of the given Orlicz function and its conjugate). In order to get operators of norm arbitrarily close to 1 we have to apply Theorem 1.

2. General results

THEOREM 1. *A superreflexive space Y which has the* u.a.p. *has the* $(1+\varepsilon)$-u.a.p. *for every* $\varepsilon > 0$.

PROOF. A superreflexive space can be renormed to be uniformly convex (cf. [4], [1]; for our purposes here we could just as well take this as the definition of superreflexivity). Moreover the uniformly convex norm can be taken to be arbitrarily close to the given norm. Indeed if $\| \ \|$ and $||| \ |||$ are equivalent norms with $||| \cdot |||$ uniformly convex then for every $\alpha > 0$, $\| \cdot \| + \alpha \, ||| \ |||$ is also uniformly convex and close to $\| \cdot \|$ for small α. In view of this remark there is no loss of generality to assume that Y is already uniformly convex with modulus of convexity $\delta(\tau)$.

Let λ_0 be the infimum of all the λ for which Y has the λ-u.a.p.; we have to show that $\lambda_0 = 1$. Assume that $\lambda_0 > 1$, put $\tau_0 = (\lambda_0 - 1)/2(\lambda_0 + 1)$ and let $\eta > 0$ be such that

$$\delta(\tau_0) > \frac{8\eta}{\eta + \lambda_0} \quad \text{and} \quad \frac{\lambda_0 - 3\eta - 1}{\lambda_0 + \eta + 1} > \tau_0 .$$

By the choice of λ_0, Y has the $(\mu_0, n(k,\mu_0))$-u.a.p. with $\mu_0 = \lambda_0 + \eta$ and a suitable function $n(k,\mu_0)$. Let E be a k-dimensional subspace of Y and let $T\colon Y \to Y$ satisfy $\| T \| \leqq \mu_0$, $T_{|E} =$ identity and $\dim TY \leqq n(k,\mu_0)$.

Set

$$K = \{y \,;\, y \in Y, \| y \| = 1, \| Ty \| \geqq \mu_0(1 - \delta(\tau_0)/2)\}.$$

Since TK is contained in a ball of radius μ_0 in an $n(k,\mu_0)$-dimensional space we can find a number $m = m(k,\mu_0)$, independent of the particular choice of E, and m balls $\{B_i\}_{i=1}^m$ of radius $\mu_0\delta(\tau_0)/4$ so that $TK \subset \bigcup_{i=1}^m B_i$. Notice that $K \subseteq \bigcup_{i=1}^m K_i$ where $K_i = K \cap T^{-1}B_i$. Assume now that there are $y', y'' \in K_i$ so that $\|y' - y''\| \geqq \tau_0$. Then

$$1 - \|y' + y''\|/2 \geqq \delta(\tau_0)$$

and, consequently,

$$\mu_0\left(1 - \frac{\delta(\tau_0)}{2}\right) \leqq \|Ty'\| \leqq \|Ty' + Ty''\|/2 + \|Ty' - Ty''\|/2$$

$$\leqq \mu_0(1 - \delta(\tau_0)) + \mu_0\delta(\tau_0)/4 = \mu_0\left(1 - \frac{3\delta(\tau_0)}{4}\right).$$

This contradiction shows that in every non-void set K_i we can select an element y_i so that $\|y - y_i\| < \tau_0$ for every $y \in K_i$. Since the subspace $F = \operatorname{span}\{E, \{y_i\}_{i=1}^m\}$ is at most $k + m(k,\mu_0)$-dimensional we can find an operator $S: Y \to Y$ such that $S_{|F} = \text{identity}$, $\|S\| \leqq \mu_0$ and $\dim SY \leqq n(k+m,\mu_0)$. Consider now the operator $\tilde{T} = (S+T)/2$. Evidently, $\tilde{T}_{|E} = \text{identity}$ and $\dim \tilde{T}Y \leqq \nu(k) = n(k,\mu_0) + n(k+m,\mu_0)$. To compute the norm of $\tilde{T}$ we shall let $y_0 \in Y$, $\|y_0\| = 1$ and distinguish between two cases. In the first we assume $y_0 \in K$. Then $y_0 \in K_i$ for some i which implies $\|y_0 - y_i\| < \tau_0$. Thus, $\|Sy_0\| \leqq \|S(y_0 - y_i)\| + \|y_i\| \leqq \mu_0\tau_0 + 1$. In view of the choice of η we have

$$\|\tilde{T}y_0\| \leqq (\|Sy_0\| + \|Ty_0\|)/2 \leqq (\mu_0 + 1)(\tau_0 + 1)/2$$

$$\leqq (\lambda_0 + \eta + 1)(\tau_0 + 1)/2 \leqq \lambda_0 - \eta.$$

In the second case, since $y_0 \notin K$, we have $\|Ty_0\| < \mu_0(1 - \delta(\tau_0)/2)$. Thus,

$$\|\tilde{T}y_0\| \leqq (\mu_0 + \mu_0(1 - \delta(\tau_0)/2))/2 = \mu_0(1 - \delta(\tau_0)/4)$$

$$= (\lambda_0 + \eta)(1 - \delta(\tau_0)/4) \leqq \lambda_0 - \eta.$$

Hence $\|\tilde{T}\| \leqq \lambda_0 - \eta$ which means that Y has the $(\lambda_0 - \eta, \nu(k))$-u.a.p. This contradicts the minimality of λ_0 and thus concludes the proof.

THEOREM 2. *A superreflexive space Y has the* u.a.p. *iff Y^* has the same property.*

PROOF. By Theorem 1 we may assume that Y is uniformly convex and has the $(1+\varepsilon, n(k,\varepsilon))$ u.a.p. for every $\varepsilon > 0$.

Fix $\varepsilon > 0$ and an integer k. Let $F \subset Y^*$ be a subspace of dimension k. There is an $m = m(k,\varepsilon)$ and a subspace $G \subset Y$ with $\dim G = m$ such that

$$(*) \qquad (1-\varepsilon)\|y^*\| < \sup_{\substack{y \in G \\ \|y\|=1}} |y^*(y)|; \qquad y^* \in F.$$

Let $T: Y \to Y$ with $T_{|G} = \text{identity}$, $\|T\| \leqq 1+\varepsilon$ and $\dim TY \leqq n(m,\varepsilon)$. Then $T^*y^*\ (y) = y^*(y)$ for every $y \in G$. Hence by (*),

$$\|T^*y^* + y^*\| \geqq 2 - 2\varepsilon$$

for every $y^* \in F$ with $\|y^*\| = 1$. Since also $\|T^*y^*\| \leqq 1+\varepsilon$ an easy calculation shows that $\|T^*y^* - y^*\| \leqq \delta^{-1}(4\varepsilon) + 2\varepsilon$ where δ is the modulus of convexity of Y^*. By a standard perturbation argument we deduce that Y^* has the u.a.p.

Remark. By using the proof of Theorem 1 and the argument in paper [2] the following can be proved. Let Y be a Banach space such that every equivalent norm in X has the 2-u.a.p. Then Y^* has the u.a.p. This shows the connection between the results of Theorem 1 and 2 even for non-superreflexive spaces. We do not know, however, whether either of the theorems is valid without the superreflexivity assumptions.

A result which can be proved without it is the following.

Theorem 3. *A Banach space Y has the* u.a.p. *if and only if Y^{**} has the* u.a.p.

The "if" part of the theorem follows directly from the local reflexivity principle. The "only if" part is a consequence of the following two propositions.

Proposition 1. *Let Y be a Banach space having the* u.a.p., *C a set of indices and U a free ultrafilter over C. Then the ultrapower Y^C/U has also the* u.a.p.

Proof. Assume that Y has the $(\lambda, n(k))$-u.a.p. and let $y^{(i)} = (y_c^{(i)})_{c \in C}$; $1 \leqq i \leqq k$ be a system of k vectors in Y^C/U. Then, for every fixed $c \in C$, there exists an operator $T_c: Y \to Y$ such that $T_c y_c^{(i)} = y_c^{(i)}$; $1 \leqq i \leqq k$; $\|T_c\| \leqq \lambda$, and $\dim T_c Y \leqq n(k)$. We shall define an operator $T: Y^C/U \to Y^C/U$ as follows: for $x = (x_c)_{c \in C}$; we set $T(x_c)_{c \in C} = (T_c x_c)_{c \in C}$. Obviously T is a linear operator satisfying $\|T\| \leqq \lambda$ and $Ty^{(i)} = y^{(i)}; 1 \leqq i < k$. To estimate the rank of T we choose, for every $c \in C$, a system of $n(k)$ unit vectors $z_c^{(i)}; 1 \leqq i \leqq n(k)$ such that $\|\sum_{i=1}^{n(k)} a^{(i)} z_c^{(i)}\| \geqq \max_{1 \leqq i \leqq n(k)} |a^{(i)}|$ and $T_c Y \subseteq \text{span}\{z_c^{(i)}\}_{i=1}^{n(k)}$ (such a system is called an Auerbach basis). Then, for any $x = (x_c)_{c \in C}$ we have

$$T_c x_c = \sum_{i=1}^{n(k)} a_c^{(i)} z_c^{(i)}$$

for some scalars $a_c^{(i)}$ satisfying $|a_c^{(i)}| \leqq \lambda \|x\|$. Consequently,

$$T_x = \left(\sum_{i=1}^{n(k)} a_c^{(i)} z_c^{(i)}\right)_{c \in C} = \sum_{i=1}^{n(k)} a^{(i)} (z_c^{(i)})_{c \in C}$$

where $a^{(i)} = \lim_u a_c^{(i)}$; $1 \leqq i \leqq n(k)$. It follows that

$$T(Y^C/U) \subset \operatorname{span}\{(z^{(i)})_{c \in C}; 1 \leqq i \leqq n(k)\} \quad \text{i.e.} \quad \dim T(Y^C/U) \leqq n(k).$$

PROPOSITION 2. *For every Banach space Y there exists an ultrapower Y^C/U and a norm one projection P in Y^C/U whose range is isometric to Y^{**}.*

PROOF. Let C be the set of all tuples (F, G, ε) where F is a finite dimensional subspace of Y^{**}, G a finite dimensional subspace of Y^* and $\varepsilon > 0$. The set C is endowed with the order $(F_1, G_1, \varepsilon_1) < (F_2, G_2, \varepsilon_2)$ iff $F_1 \subset F_2$, $G_1 \subset G_2$ and $\varepsilon_1 > \varepsilon_2$ becomes a directed set.

Let U be a free ultrafilter on C which is consistent with this order on C. By the local reflexivity principle, for any such triplet $c = (F, G, \varepsilon) \in C$, there exists an operator $S_c : F \xrightarrow{\text{into}} Y$ such that $S_{c|F \cap Y} =$ identity, $\|S_c\| \cdot \|S_c^{-1}\| < 1 + \varepsilon$, and $y^*(S_c y^{**}) = y^{**}y^*$ for every $y^{**} \in F$ and every $y^* \in G$. For $y^{**} \in Y^{**}$ we can now set

$$\tilde{S}_c y^{**} = \begin{cases} S_c y^{**} & \text{if } y^{**} \in F \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be easily verified that $Sy^{**} = (\tilde{S}_c y^{**})_{c \in C}$ defines an isometry S from Y^{**} into Y^C/U.

We can also define an operator T from Y^C/U into Y^{**} by setting

$$(T(y_c)_{c \in C})(y^*) = \lim_U y^* y_c;\ y^* \in Y^*;\ (y_c)_{c \in C} \in Y^C/U.$$

For $c_0 = (F_0, G_0, \varepsilon_0)$ and $y^{**} \in F_0$, $y^* \in G_0$ we have

$$(TSy^{**})(y^*) = \lim_U y^* \tilde{S}_c y^{**} = \lim_U y^* S_c y^{**} = y^{**}y^*$$

which shows that $TS =$ identity on Y^{**}. If we also notice that $\|T\| \leqq 1$ then $P = ST$ is the desired projection.

Proposition 2 and its proof are due to J. Stern [12].

We conclude this section by stating the following Corollary of Theorem 3.

COROLLARY. *Let Y have the* u.a.p. *Then all the conjugates of Y have the approximation property.*

PROOF. Use Theorem 3 and a result of Grothendieck [3] which states that if for some Banach space X the dual X^* has the approximation property then the same is true for X.

3. Orlicz spaces

Before we pass to the study of the u.a.p. in Orlicz spaces we recall a result concerning the u.a.p. in general Banach lattices (or equivalently in spaces with an unconditional basis). This result shows that in studying the u.a.p. in lattices it is enough to consider finite dimensional subspaces E of lattices L spanned by disjointly supported elements. In the setting of L_p spaces this was proved in [11]; however, the same argument works in the general setting (this was pointed out to us by W. B. Johnson).

PROPOSITION 3. *There exists a function* $N(k,\varepsilon)\,(N(k,\varepsilon)=[2k^2/\varepsilon]^k)$ *such that for any fixed* $\varepsilon>0$, *every Banach lattice L and every k-dimensional subspace F of L there are* $N=N(k,\varepsilon)$ *disjoint elements* $\{g_j\}_{j=1}^N$ *in L and a linear operator* $V\colon F\xrightarrow{\text{into}} G=\operatorname{span}\{g_j\}_{j=1}^N$ *so that* $\|Vf-f\|\leqq\varepsilon\|f\|$ *for all* $f\in F$.

PROOF. Let $\dim F=k$ and let $\{f_i\}_{i=1}^k$ be an Auerbach basis in F (i.e. $\|\Sigma_{i=1}^k a_i f_i\|\geqq\max|a_i|$ for all choices of $\{a_i\}_{i=1}^k$). Set $f_0=\Sigma_{i=1}^k|f_i|/k$ where $|\cdot|$ denotes the absolute value in L i.e. $|f|=f\vee(-f)$. Let Z be the sublattice of all $f\in L$ for which there exists some $t>0$ so that $|f|<tf_0$. Then $|||f|||=\inf\{t>0;|f|<tf_0\}$ is a norm in Z and Z endowed with this norm is an abstract M-space with a unit (namely f_0).

Let $f=\Sigma_{i=1}^k a_i f_i$ be an element of norm 1 in F. Then $|f|\leqq\Sigma_i|a_i|\,|f_i|\leqq kf_0$ and hence $|||f|||\leqq k$. The unit ball in F is thus contained in a ball of radius k in $(Z,|||\ |||)$. The proof of Proposition 3 in the case of L_∞ (cf. [11]), which by Kakutani's theorem applies also in $(Z,|||\ |||)$, shows that there are $N=[2k^2/\varepsilon]^k$ elements $\{g_j\}_{j=1}^N$ in Z and an operator $V\colon F\xrightarrow{\text{into}} G=\operatorname{span}\{g_j\}$ such that $|||Vf-f|||<\varepsilon$ for every $f\in F$ with $|||f|||\leqq k$. Hence $|Vf-f|<\varepsilon f_0$ and thus also $\|Vf-f\|\leqq\varepsilon$ for every $f\in F$ with $\|f\|=1$. This completes the proof.

REMARKS. 1. The argument we presented here is an obvious modification of an argument due to Kwapien (and presented in [11]) who showed how to reduce the proof of the proposition in the case $L=L_p(0,1)$ to the simplest case i.e.

$p = \infty$. 2. Since in an L_p-space the span of disjointly supported elements is the range of a contractive projection Proposition 3 shows that the L_p spaces have the uniform projection property. This argument works only for L_p-space.

We pass now to Orlicz spaces. We shall work in the setting of reflexive Orlicz sequence spaces l_M. The proof that these spaces have the u.a.p. is based on the existence of a large supply of disjoint blocks whose spans are complemented in the space. We want first to explain this point in order to clarify the computations done below. Assume that we are given blocks $g_j = \Sigma_{s\in\sigma_j} t_s e_s$ where the σ_j are disjoint finite subsets of the integers and $\{e_s\}_{s=1}^{\infty}$ denotes the canonical unit vector basis in l_M. Assume that for each j there is a function $N_j(x)$ so that $N_j(x) = M(t_s x)/M(t_s)$ for all $s \in \sigma_j$ and every $x \in [0,1]$. Then the span of the $\{g_j\}$ is the range of a contractive projection from l_M. The projection is a weighted averaging projection and is given by

$$Pf = \sum_j \left(\sum_{i\in\sigma_j} x_i M(t_i/\| g_j \|)/t_i \right) g_j .$$

We omit the easy verification that P is a contractive projection; this will enter into the proof presented below. For a reflexive Orlicz space the set $E_{M,1} = \{M(tx)/M(t)\}_{0<t<1}$ is a compact subset of $C(0,1)$. Hence, given any finite number of disjoint blocks in l_M and an $\varepsilon > 0$ we can subdivide the blocks into a finite number of smaller blocks such that in each of the small blocks say η the functions $\{M(t_i x)/M(t_i)\}_{i\in\eta}$, while not identical, form a set of diameter $\leqq \varepsilon$ in $C(0,1)$. Under certain assumptions the projection P (more precisely a variant of it) defined above (corresponding to the small blocks) will still work. Into the evaluation of the norm of P enters in a crucial way the number of points in a ε-net of the compact set $E_{M,1}$. It turns out that we can ensure that $\| P \| \leqq \lambda$ where λ is a constant depending only on the space. The technical part of the computation is somewhat simplified if we use a representation of Orlicz functions by sequences of 0's and 1's (introduced in [8]). We recall briefly this representation.

Let l_M be a reflexive Orlicz sequence space. Because of the reflexivity we can assume with no loss of generality that for some $1 < p < r$ and all $0 < x \leqq 1$ we have $p \leqq xM'(x)/M(x) \leqq r$. Let α be the (unique) number satisfying $\alpha p - p + 1 = \alpha^r$. Then the functions $F(x) = x^p$ and $G(x) = px - p + 1$ have on the interval $[\alpha, 1]$ the following properties:

$$F(1) = G(1) = 1; F(\alpha) = \alpha^p; G(\alpha) = \alpha^r$$

$$xF'(x)/F(x) \geqq F'(1) = p = G'(1) \leqq xG'(x)/G(x); \alpha \leqq x \leqq 1 .$$

Under these conditions (cf. [6]), for any sequence $\theta=\{\theta(i)\}_{i=1}^{\infty}$, where $\theta(i)$ is either 0 or 1, we can define an Orlicz function M_θ on $[0,1]$ by setting:

$$M_\theta(0)=0;\ M_\theta(1)=1$$
$$M_\theta(x)=\begin{cases} M_\theta(\alpha^{i-1})F(x/\alpha^{i-1}) & \text{if } \theta(i)=0 \\ M_\theta(\alpha^{i-1})G(x/\alpha^{i-1}) & \text{if } \theta(i)=1 \end{cases} \quad \begin{matrix} \alpha^i \leqq x < \alpha^{i-1} \\ i=1,2,\cdots. \end{matrix}$$

It can be easily checked that $p\leqq xM'_\theta(x)/M_\theta(x)\leqq r; 0<x\leqq 1$.

In order to get a function M_θ which is equivalent to M we shall define the sequence $\{\theta(i)\}_{i=1}^{\infty}$ in the following inductive way: we put $\theta(1)=1$ and if $M_\theta(\alpha^i)\alpha^p\leqq M(\alpha^{i+1})$ then we set $\theta(i+1)=0$; otherwise we take $\theta(i+1)=1$. Since $\alpha^r\leqq M(\alpha x)/M(x)\leqq\alpha^p$; $0<x\leqq 1$ we can verify that $M_\theta(\alpha^i)\leqq M(\alpha^i)\leqq \alpha^{p-r}M_\theta(\alpha^i)$; $i=1,2,\cdots$ i.e. M_θ is equivalent to M.

An important remark about this construction is that α can be chosen to be as small as desired by fixing r and taking p sufficiently close to 1.

THEOREM 4. *Every reflexive Orlicz sequence space has the* $1+\varepsilon-$u.a.p., *for all* $\varepsilon>0$.

PROOF. Since reflexive Orlicz spaces are uniformly convex (cf. [9], [10]), in view of Theorem 1 it suffices to show that the space has the λ-u.a.p. for some $\lambda>1$.

Let then l_M be a reflexive Orlicz sequence space. As explained above we can find numbers $1<p<r$, $0<\alpha<1$ and a sequence $\theta=\{\theta(i)\}_{i=1}^{\infty}$, with $\theta(i)$ being equal to 0 or 1, so that the Orlicz function M_θ, defined above, is equivalent to M. We have also noticed that by changing p we can choose α as small as desired. We shall assume that $\alpha<4^{-r}$.

For simplicity we shall write M instead of M_θ. It is also clear that we still have $p\leqq xM'(x)/M(x)\leqq r$; $0<x\leqq 1$.

Now let E be a k-dimensional subspace of l_M and fix $\varepsilon>0$. By Proposition 3 we can find $N=N(k,\varepsilon)$ normalized disjoint blocks $f_j=\Sigma_{i\in\sigma_j}t_ie_i$; $t_i\neq 0$ for $i\in\sigma_j$; $j=1,2,\cdots,N$ so that $F=\operatorname{span}\{f_j\}_{j=1}^{N}$ contains E up to ε (here, as usual, $\{e_i\}$ denotes the unit vector basis of l_M). A simple perturbation argument shows that it suffices to prove the u.a.p. for F.

Since $2^m\alpha^n<2^m\cdot 4^{-m}\to 0$ as $n\to\infty$ we can choose an integer $Q=Q(N)$ so that

$$N^{2r+1}2^{rQ}\alpha^Q\leqq 1.$$

Notice that there are at most 2^Q distinct sequences among those having the form $\{\theta(m+1),\theta(m+2),\cdots,\theta(m+Q)\}$; $m=0,1,2,\cdots$. Thus it is always possible to find 2^Q integers $m_1, m_2, \cdots, m_{2^Q}$ so that for every m there is a $1 \leqq \nu \leqq 2^Q$ for which $\theta(m+i) = \theta(m_\nu + i)$; $i = 1, 2, \cdots, Q$.

It is easy to see that, in the same way in which $M = M_\theta$ corresponds to $\{\theta(i)\}_{i=1}^{\infty}$, the function $M(\alpha^m x)/M(\alpha^m)$ corresponds to the sequence $\{\theta(i + m)\}_{i=1}^{\infty}$. Hence, by the definition of M_θ we have

$$\frac{M(\alpha^m x)}{M(\alpha^m)} = \frac{M(\alpha^{m_\nu} x)}{M(\alpha^{m_\nu})}; \qquad \alpha^Q \leqq x \leqq 1$$

and, therefore,

$$\left| \frac{M(\alpha^m x)}{M(\alpha^m)} - \frac{M(\alpha^{m_\nu} x)}{M(\alpha^{m_\nu})} \right| \leqq \alpha^Q; \qquad 0 \leqq x \leqq 1.$$

In the next step we shall split the set σ_j; $j = 1, 2, \cdots, N$ into disjoint subsets as follows. First, we shall denote by δ_j the set of all $i \in \sigma_j$ for which $|t_i| > N^{-2}2^{-Q-1}$; then we shall split $\sigma_j - \delta_j$, i.e. those indices $i \in \sigma_j$ for which t_i is relatively small, into disjoint subsets $\sigma_{j1}, \sigma_{j2}, \cdots, \sigma_{jh_j}$ so that h_j is *maximal*,

$$N^{-2}2^{-Q} \leqq \left\| \sum_{i \in \sigma_{jh}} t_i e_i \right\| < N^{-2}2^{-Q+1}$$

and there exists an index $\nu = \nu(j, h)$, common to all $i \in \sigma_{jh}$ in the sense that whenever $m(i)$ satisfies $\alpha^{m(i)+1} < |t_i| N^2 2^Q \leqq \alpha^{m(i)}$ then

$$\left| \frac{M(\alpha^{m(i)} x)}{M(\alpha^{m(i)})} - \frac{M(\alpha^{m_\nu} x)}{M(\alpha^{m_\nu})} \right| \leqq \alpha^Q; \qquad 0 \leqq x \leqq 1.$$

Notice that in general $\sigma_{j0} = (\sigma_j - \delta_j) - \bigcup_{h=1}^{h_j} \sigma_{jh} \neq \emptyset$ but

$$\left\| \sum_{i \in \sigma_{j0}} t_i e_i \right\| < 2^Q \cdot N^{-2}2^{-Q} = N^{-2}; \qquad j = 1, 2, \cdots, N$$

since for each possible ν; $1 \leqq \nu \leqq 2^Q$ the norm of that portion of $\sum_{i \in \sigma_j - \delta_j} t_i e_i$ which has not been accounted for in any of the sets σ_{jh}; $h = 1, 2, \cdots, h_j$ and which "corresponds" to ν is not greater than $N^{-2}2^{-Q}$. From now on we assume as we clearly may that $t_i \geqq 0$ for all i.

We are now prepared to define a finite rank operator $T: l_M \to l_M$ so that $Tf_j = f_j$; $j = 1, 2, \cdots, N$. We first choose a functional $f_j^* \in l_M^*$ which is supported by the same indices as f_j and which satisfies $\|f_j^*\| = 1$ and $f_j^*(f_j) = 1$; $j = 1, 2, \cdots, N$. Then, for every $x = \sum_{s=1}^{\infty} x_s e_s$ we set

$$Tx = \sum_{j=1}^{N} \Big\{ \sum_{s\in\delta_j} x_s e_s + f_j^*\Big(\sum_{s\in\sigma_j} x_s e_s\Big) \sum_{i\in\sigma_{j0}} t_i e_i$$

$$+ \sum_{h=1}^{h_j} \Big[\sum_{j\in\sigma_{jh}} M\Big(t_s \Big/ \Big\| \sum_{\nu\in\sigma_{jh}} t_\nu e_\nu \Big\|\Big)\frac{x_s}{t_s}\Big] \sum_{i\in\sigma_{jh}} t_i e_i \Big\}$$

Obviously, T is well defined, linear and $Tf_j = f_j$; $j = 1,2,\cdots,N$ since

$$\sum_{s\in\sigma_{jh}} M\Big(t_s \Big/ \Big\| \sum_{\nu\in\sigma_{jh}} t_\nu e_\nu \Big\|\Big) = 1.$$

To estimate the dimension of the range of T we need the fact that for any sequence of disjoint blocks $\{u_j\}$ in l_M we have $\|\Sigma_j u_j\| \geqq (\Sigma_j \|u_j\|^r)^{1/r}$. Using this inequality (which follows easily from $xM'(x)/M(x) \leqq r$; $0 < x \leqq 1$ and the correspondence between blocks in l_M and Orlicz functions in $C_{M,1} = \operatorname{conv}\{M(tx)/M(t);\, 0 < t \leqq 1\}$; for more details see [6]) we get that δ_j contains at most $N^{2r}\cdot 2^{(Q+1)r}$ elements. Similarly, it follows that $h_j \leqq N^{2r}\cdot 2^{Qr}$. Thus, $\dim Tl_M \leqq N[N^{2r}\cdot 2^{(Q+1)r} + 1 + N^{2r}\cdot 2^{Qr}] \leqq N^{2r+1}\cdot 2^{2(Q+1)r}$ where $N = N(k,\varepsilon)$ and $Q = Q(N)$.

To estimate the norm of T we assume that $x_j \geqq 0$ and $\|x\| \leqq 1$ i.e. $\Sigma_{s=1}^{\infty} M(x_s) \leqq 1$. We first notice that

$$\Big\| \sum_{j=1}^{N} f_j^*\Big(\sum_{s\in\sigma_j} x_j e_j\Big) \sum_{i\in\sigma_{j0}} t_i e_i \Big\| \leqq N\cdot N^{-2} = N^{-1}.$$

Now fix j and h. Then, by the convexity of M and the fact that $M(\gamma x)/M(x) \leqq \gamma^r$ for any $\gamma > 1$ and $0 < x \leqq 1$, we have:

$$A_{jh} = \sum_{i\in\sigma_{jh}} M\Big(t_i \sum_{s\in\sigma_{jh}} M\Big(t_s \Big/ \Big\| \sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\Big)\frac{x_s}{t_s}\Big)$$

$$\leqq 2^{r-1} \sum_{i\in\sigma_{jh}} M\Big(t_i {\sum_s}' M\Big(t_s \Big/ \Big\| \sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\Big)\frac{x_s}{t_s}\Big)$$

$$+ 2^{r-1} \sum_{i\in\sigma_{jh}} M\Big(t_i {\sum_s}'' M\Big(t_s \Big/ \Big\| \sum_{\nu\in r_{jh}} t_\nu e_\nu\Big\|\frac{x_s}{t_s}\Big)$$

where Σ' contains all the indices s for which $t_s/\|\Sigma_{\nu\in\sigma_{jh}} t_\nu e_\nu\| \leqq x_s$ and Σ'' the others. By the convexity of M it follows that $M(x)/x$ is an increasing function. Hence,

$$\sum_{i\in\sigma_{jh}} M\Big(t_i {\sum_s}' M\Big(t_s \Big/ \Big\| \sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\Big)\frac{x_s}{t_s}\Big) \leqq \sum_{i\in\sigma_{jh}} M\Big(t_i {\sum_s}' M(x_s) \Big/ \Big\| \sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\Big)$$

$$\leqq \sum_{i\in\sigma_{jh}} {\sum_s}' M(x_s) M\Big(t_i \Big/ \Big\| \sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\Big) = {\sum_s}' M(x_s).$$

On the other hand, again by convexity of M, we have

$$\sum_{i\in\sigma_{jh}} M\left(t_i \sum_s{}'' M\left(t_s\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right)\frac{x_s}{t_s}\right)$$

$$\leqq \sum_{i\in\sigma_{jh}} \sum_s{}'' M\left(t_s\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right) M(t_i x_j/t_s).$$

Furthermore, for $\nu=\nu(j,h)$, $i\in\sigma_{jh}$ and s corresponding to Σ'' we have

$$\begin{aligned}
M(t_i x_s/t_s) &\leqq M\left(t_i N^2\cdot 2^Q\cdot x_s\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\Big/t_s\right)\\
&\leqq M\left(\alpha^{m(i)}x_s\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\Big/t_s\right)\\
&\leqq M(\alpha^{m(i)})\left[\alpha^Q + M\left(\alpha^{m_\nu}x_s\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\Big/t_s\right)\Big/M(\alpha^{m_\nu})\right]\\
&\leqq (2/\alpha)^r M\left(t_i\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right)\left[2\alpha^Q + M\left(\alpha^{m(s)}x_s\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\Big/t_s\right)\Big/M(\alpha^{m(s)})\right]\\
&\leqq (2/\alpha)^r M\left(t_i\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right)\left[2\alpha^Q + (2/\alpha)^r M(x_s)/M\left(t_s\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
A_{jh} &\leqq 2^{r-1}\sum_s{}' M(x_s) + 2^{r-1}\sum_{i\in\sigma_{jh}}\sum_s{}'' M\left(t_s\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right)\\
&\quad\cdot 2^{r+1}\cdot\alpha^{Q-r} M\left(t_i\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right)\\
&\quad + 2^{r-1}\sum_{i\in\sigma_{jh}}\sum_s{}'' M\left(t_s\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right)2^{2r}\\
&\quad\cdot\alpha^{-2r}M(x_s)\,M\left(t_i\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right)\Big/M\left(t_s\Big/\Big\|\sum_{\nu\in\sigma_{jh}} t_\nu e_\nu\Big\|\right)\\
&\leqq 2^{r-1}\sum_s{}' M(x_s) + 2^{2r}\cdot\alpha^{Q-r} + 2^{3r-1}\alpha^{-2r}\sum_s{}'' M(x_s)\\
&\leqq 2^{3r-1}\cdot\alpha^{-2r}\sum_{s\in\sigma_{jh}} M(x_s) + 2^{2r}\alpha^{Q-r}.
\end{aligned}$$

It follows from this that

$$\sum_{j=1}^{N}\left\{\sum_{s\in\delta_j} M(x_s)+\sum_{h=1}^{h_j} A_{jh}\right\} \leqq 2^{3r-1}\cdot\alpha^{-2r}\sum_{s=1}^{\infty} M(x_s)$$

$$+2^{2r}\cdot\alpha^{Q-r}\sum_{j=1}^{N} h_j \leqq 2^{3r-1}\cdot\alpha^{-2r}\sum_{s=1}^{\infty} M(x_s)$$

$$+2^{2r}\cdot\alpha^{Q-r}N\cdot N^{2r}\cdot 2^{Qr} \leqq 2^{3r-1}\alpha^{-2r}+2^{2r}\alpha^{-r}\leqq 2^{3r}\alpha^{-2r}.$$

Thus $\| T\| \leqq N^{-1}+2^{3r}\cdot\alpha^{-2r}<(2/\alpha)^{3r}$ and this completes the proof.

REMARKS. 1. The bounds for the norm of T and for $\dim Tl_M$ are the same for all the Orlicz functions M for which $1<p\leqq xM'(x)/M(x)\leqq r;\ 0<x\leqq 1$ with the same p and r. This is equivalent to the fact that reflexive Orlicz spaces l_M have the u.a.p. with constants depending only on the Δ_2-constants of M and its dual functions M^*. 2. The operator T defined above acts as a projection on $\operatorname{span}\{e_i;\, i\notin \bigcup_{j=1}^{N}\sigma_{j0}\}$. Since the norm of T restricted to $\operatorname{span}\{e_i;\, i\in\bigcup_{j=1}^{N}\sigma_{j0}\}$ is less than N^{-1} we can apply a simple perturbation argument and replace T by a projection P in l_M so that $Pf_j=f_j;\, j=1,2,\cdots,N;$ $\|P\|\leqq(2/\alpha)^{3r}+1,$ and $\dim Pl_M=\dim Tl_M$. This means that reflexive Orlicz sequence space have even the uniform projection property. 3. Reflexive modular sequence spaces can always be embedded as complemented subspaces of reflexive Orlicz sequence spaces (this follows from the construction of universal Orlicz functions presented in [7]). Hence, they also have the $1+\varepsilon-$u.a.p. for all $\varepsilon>0$.

COROLLARY. *Every reflexive Orlicz function space L_M, on either a finite or infinite interval, has the* $1+\varepsilon-$u.a.p. *for all* $\varepsilon>0$.

PROOF. Since the u.a.p. is a local property we have to consider only the case of Orlicz spaces $L_M(a,b)$ where (a,b) is a finite interval. Let E be an h-dimensional subspace of $L_M(a,b)$. Then, for any $\varepsilon>0$ we can find a number c and an operator $V\colon E\xrightarrow{\text{into}} H=\operatorname{span}_{1\leqq j\leqq q=(b-a)/c}\{\chi_{[a+(j-1)c,\,a+jc]}\}$ so that $\|Ve-e\|<\varepsilon;\ e\in E$. Evidently, c can and should be chosen in such a manner that q is an integer. The number q can be very large and certainly depends on E. However, H is actually isometric to the Orlicz sequence space l_N^q where $N(x)=M(dx)/M(d)$ and d is defined by $cM(d)=1$.

Since H has the u.a.p. with constants depending only on M and since H is the range of a contractive projection in L_M it follows immediately that L_M also has the u.a.p. and therefore the $1+\varepsilon-$u.a.p. for all $\varepsilon>0$.

REFERENCES

1. P. Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Israel J. Math. **13** (1972), 281–288.

2. T. Figiel and W. B. Johnson, *The approximation property does not imply the bounded approximation property*, Proc. Amer. Math. Soc. **41** (1973), 197–200.

3. A. Grothendieck, *Produits tensoriels topologiques et espaces nucleaires*, Mem. Amer. Math. Soc. **16** (1955).

4. R. C. James, *Some self-dual properties of normed linear spaces*, Symposium on Infinite Dimensioned Topology, Ann. of Math. Studies **69** (1971).

5. J. Lindenstrauss, *On the modulus of smoothness and divergent series in Banach spaces*, Michigan Math. J. **10** (1963), 241–252.

6. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math. **10** (1971), 379–390.

7. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces II*, Israel J. Math. **11** (1972), 355[379.

8. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces III*, Israel J. Math. **14** (1973), 368–389.

9. W. A. J. Luxemburg, *Banach function spaces*, Thesis, Netherlands, 1955.

10. H. W. Milnes, *Convexity of Orlicz spaces*, Pacific J. Math. **7** (1957), 1451–1483.

11. A. Pelczynski and H. P. Rosenthal, *Localization techniques in L^p-spaces*, Studia Math. **52** (1975), 263–289.

12. J. Stern, *Propriétés locales et ultrapuissances d'espace de Banach*, Seminaire Maurey-Schwartz, 1974–75, Exposés VII–VIII.

13. A. Szankowski, *A Banach lattice without approximation property*, to appear.

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